Some fuzzifications in ordered AG**-groupoid

Faisal, Kifayat Ullah, Abdul Haseeb, Naveed Yaqoob
Department of Mathematics, COMSATS Institute of Information Technology Abbottabad, K. P. K., Pakistan
Department of Mathematics, Kohat University of Science & Technology, Kohat, K. P. K., Pakistan
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
E-mail: {youfazaimath@yahoo.com, haseeb_yuz@yahoo.com, kifayatmath@yahoo.com, nayaqoob@ymail.com}

ABSTRACT

We have given the concept of fuzzy ordered AG**-groupoids and studied some important structural properties of a (2,2) regular ordered AG**-groupoid in terms of different fuzzy ideals.

Keywords: Ordered AG**-groupoid, (2,2) regular ordered AG**-groupoid, left invertive law and fuzzy ideals.

1. INTRODUCTION

The concept of fuzzy set was first proposed by Zadeh [14] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It gives us a tool to model the uncertainty present in phenomena that do not have sharp boundaries. Many papers on fuzzy sets have been appeared which shows the importance and its applications to set theory, algebra, real analysis, measure theory and topology etc.

The concept of an Abel Grassmann's groupoid, abbreviated as an AG-groupoid was first introduced by Kazim and Naseeruddin in 1972 [2] and they have called it a left almost semi group (LA-semi group). An AG-groupoid may or may not contain a left identity. The left identity of an AG-groupoid allows us to introduce the inverses of elements in an AG-groupoid.

An AG-groupoid [10] is a groupoid \( S \) holding the left invertive law

\[
(ab)c=(cb)a, \text{ for all } a,b,c \in S \tag{1}
\]

This left invertive law has been obtained by introducing braces on the left of ternary commutative law \((ab)c=(cb)a\).

In an AG-groupoid, the medial law holds [2]

\[
(ab)(cd)=(ac)(bd), \text{ for all } a,b,c,d \in S \tag{2}
\]

If an AG-groupoid contains a left identity, then it is unique [7]. In an AG-groupoid \( S \) with left identity [7], the paramedial law holds

\[
(ab)(cd)=(dc)(ba), \text{ for all } a,b,c,d \in S \tag{3}
\]

Further if an AG-groupoid contains a left identity, the following law holds [7]

\[
a(bc)=b(ac), \text{ for all } a,b,c \in S \tag{4}
\]

If an AG-groupoid (without left identity) satisfies (3), then it is called an AG**-groupoid. An AG**-groupoid also satisfies (4).

An AG**-groupoid is the generalization of an AG-groupoid with left identity. Every AG-groupoid with left identity is an AG**-groupoid but the converse is not true in general. Let us consider an AG-groupoid with a binary operation \(*\) defined in the following Cayley table.

\[
\begin{array}{ccc}
* & 1 & 2 & 3 \\
1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
3 & 1 & 2 & 2 \\
\end{array}
\]

It is easy to see that the above AG-groupoid is an AG**-groupoid but it does not contain a left identity.

An AG-groupoid is a non-associative and non-commutative algebraic structure midway between a groupoid and a commutative semi group. This structure is closely related with a commutative semi group, because if an AG-groupoid contains a right identity, then it becomes a commutative semi group [7]. The connection of a commutative inverse semi group with an AG-groupoid has been given in [8] as, a commutative inverse semi group \((S,\circ)\) becomes an AG-groupoid \((S,\cdot)\) under \(a \cdot b = b \circ a^{-1}\) for all \(a,b \in S\). An AG-groupoid \( S \) with left identity becomes a semi group under the binary operation "\(*\)" defined as, if for all \(x,y \in S\), there exists \(a \in S\) such that \(x \circ y = (xa)y\) [11]. An AG-groupoid is the generalization of a semi group theory [7] and has vast applications in collaboration with semi group like other branches of mathematics. The connection of AG-groupoids with the vector spaces over finite fields has been investigated in [3].

From the above discussion, we see that AG-groupoids have very close links with semigroups and vector spaces which shows the importance and
The concept of fuzzy ordered AG-groupoid was first given by Khan and Faisal in [4]. In this paper, we have given the concept of fuzzy ordered AG**-groupoid. An ordered AG**-groupoid (po-AG**-groupoid) is a structure \((S, \leq, \cdot, \rhd)\) in which the following conditions hold.

(i) \((S, \leq)\) is an AG**-groupoid.

(ii) For all \(a, b, x \in S\), \(a \leq b \Rightarrow ax \leq bx\) and \(xa \leq x b\).

Example 1: Consider an open interval \(R_0 = (0, 1)\) of real numbers under the binary operation of multiplication. Define \(a \cdot b = a + b - ab\) for all \(a, b \in R_0\), then it is easy to see that \((R_0, \cdot, \leq)\) is an ordered AG**-groupoid under the usual order \(\leq\). This is a natural example of an ordered AG**-groupoid called a real ordered AG**-groupoid.

Basic Definitions and Results

In this section, we have given some basic definitions which are necessary for the subsequent sections. Throughout in this paper \(S\) will be considered as an ordered AG**-groupoid and \(F(S)\) will denote all the fuzzy subsets of \(S\) unless otherwise specified.

A fuzzy subset or a fuzzy set of a non-empty set \(S\) is an arbitrary mapping \(f: S \rightarrow [0, 1]\) where \([0, 1]\) is the unit segment of real line. The product of any fuzzy subsets \(f\) and \(g\) of \(S\) is defined by

\[
(f \circ g)(x) = \left\{ \begin{array}{ll}
V \{y, z \in S \mid f(y) \wedge g(z) \}
\end{array} \right.
\]

if \(x \leq yz\) \((Ax \neq \emptyset)\), and

\[
(f \circ g)(x) = \left\{ \begin{array}{ll}
0, \quad \text{if} \ x \not\leq yz \ (Ax = \emptyset)
\end{array} \right.
\]

The order relation \(\leq\) between any two fuzzy subsets \(f\) and \(g\) of \(S\) is defined by \(f \leq g\) iff \(f(x) \leq g(x)\), for all \(x \in S\).

The symbols \(f \cap g\) and \(f \cup g\) will mean the following fuzzy subsets of \(S\)

\[
(f \cap g)(x) = f(x) \wedge g(x), \text{ for all } x \in S
\]

and

\[
(f \cup g)(x) = f(x) \vee g(x), \text{ for all } x \in S
\]

For \(\emptyset \neq A \subseteq S\), we define

\[
(A) = \{ t \in S \mid t \leq a, \text{ for some } a \in A\}
\]

For \(A = \{a\}\), we usually written as \(a\).

A subset \(A\) of \(S\) is called semiprime if \(a^2 \in A\) implies \(a \in A\). A fuzzy subset \(f\) of \(S\) is called a fuzzy semiprime if \(f(a) \geq f(a^2)\), for all \(a \in S\).

A non-empty subset \(A\) of \(S\) is called a left (right) ideal of \(S\) if

(i) \(S \subseteq A \subseteq (A \subseteq S)\).

(ii) If \(a \in A\) and \(b \in S\) such that \(b \leq a\), then \(b \in A\).

A non-empty subset \(A\) of \(S\) is called a two-sided ideal of \(S\) if it is both a left and a right ideal of \(S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy left (right) ideal of \(S\) if

(i) \(x \leq y \Rightarrow f(x) \geq f(y)\), for all \(x, y \in S\).

(ii) \(f(ab) \geq f(b(f(a) + f(a)))\) for all \(a, b \in S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy two-sided ideal of \(S\) if it is both a fuzzy left and a fuzzy right ideal of \(S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy generalized bi-ideal of \(S\) if

(i) \(x \leq y \Rightarrow f(x) \geq f(y)\), for all \(x, y \in S\).

(ii) \(f((xy)z) \geq f(x) \wedge f(z)\), for all \(x, y, z \in S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy bi-ideal of \(S\) if

(i) \(x \leq y \Rightarrow f(x) \geq f(y)\), for all \(x, y \in S\).

(ii) \(f((xy)z) \geq f(x) \wedge f(z)\), for all \(x, y, z \in S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy interior ideal of \(S\) if

(i) \(x \leq y \Rightarrow f(x) \geq f(y)\), for all \(x, y \in S\).

(ii) \(f((xy)z) \geq f(x) \wedge f(z)\), for all \(x, y, z \in S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy (1,2)-ideal of \(S\) if

(i) \(x \leq y \Rightarrow f(x) \geq f(y)\), for all \(x, y \in S\).

(ii) \(f((xy)z) \geq f(x) \wedge f(z)\), for all \(x, y, z \in S\).

A fuzzy subset \(f\) of \(S\) is a fuzzy groupoid if \(f \circ f = f\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy AG**-subgroupoid of \(S\) if \(f(a) = f(b) \wedge f(c) = f(d) = f(e) = 0.1\), then it is easy to see that \(f\) is a fuzzy AG**-subgroupoid of \(S\).

A fuzzy subset \(f\) of \(S\) is called a fuzzy AG**-subgroupoid of \(S\) if \(f(ab) = f(b) \wedge f(c) = f(d) = f(e) = 0.1\), then it is easy to see that \(f\) is a fuzzy AG**-subgroupoid of \(S\).

Indeed \(f(bd) = f(d) \wedge f(db) = f(f(d)))\).
The following Proposition has the same proof as in [4].

**Proposition:** The set $(F(S), \circ, \subseteq)$ satisfies the following properties.

(i) $(F(S), \circ, \subseteq)$ is an ordered AG**-groupoid.

(ii) In $(F(S), \circ, \subseteq)$, (i) and (2) hold.

**Theorem 1:** In $S$, the following properties hold.

(i) $f \circ (g \circ h) = (f \circ g) \circ (g \circ k)$, for all $f, g, h, k \in F(S)$.

(ii) $f \circ (g \circ h) = g \circ (f \circ h)$, for all $f, g, h$ in $F(S)$.

**Proof:**

(i): Assume that $Ax=\emptyset$ for any $x \in S$, then

$$(f \circ g)(x) = (f \circ (g \circ k))(x) = \bigvee \{f(y) \in Ax \mid (y, z) \in S \}$$

for all $f, g, h, k \in F(S)$. Let $Ax \neq \emptyset$, then there exist $y$ and $z$ in $S$ such that $(y, z) \in Ax$. Therefore by using (4), we have

$$(f \circ g)(x) = \bigvee \{f(y) \in Ax \mid (y, z) \in S \}$$

for all $f, g, h, k \in F(S)$.

(ii): It is simple. Theorem gives the analogous definitions of (3) and (4) in fuzzy context.

[4] **Lemma 1:** Let $f$ be a fuzzy subset of an ordered AG**-groupoid $S$, then $f$ is a fuzzy left (right) ideal of $S$ if and only if $f$ satisfies the following.

(i) $x \leq y \Rightarrow f(x) \geq f(y)$ for all $x, y \in S$.

(ii) $S \circ f \subseteq f (S \circ f \subseteq f)$.

**Semilattice structure**

In this section, we have proved that the set of all fuzzy two-sided ideals of a (2,2) regular ordered AG**-groupoid $S$ forms a semilattice structure with identity $S$.

**Example 4:** Let $S\{a, b, c, d, e\}$ be an ordered AG**-groupoid defined in the following multiplication table and order below.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
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<td>b</td>
<td>b</td>
<td>b</td>
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<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>b</td>
<td>e</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>b</td>
<td>d</td>
<td>e</td>
<td>c</td>
</tr>
</tbody>
</table>

It is easy to see that $S$ is a (2,2) regular. Indeed for each $x \in S$ there exists $y \in S$ such that $a \leq (x^2)yx^2$.

**Lemma 2:** In a (2,2) regular $S$, for $f \circ S = f$ and $S \circ f = f$ holds for every fuzzy two-sided ideal $f$ of $S$.

**Proof:** Let $f$ be a fuzzy two-sided ideal of a (2,2) regular $S$, then for every $a \in S$ there exists $y \in S$ such that $a \leq (a^2) \leq a^2$. Now by using (5), (4) and (1), we have

$a \leq (a^2) \leq a^2$

for every fuzzy two-sided ideal $f$ of $S$. Thus $((ba)a), a) \in Aa$, since $Aa \neq \emptyset$, therefore

$(f \circ S)(a) = \bigvee \{((ba)a, a) \in Aa \mid f((ba)a \wedge S(a))\} \\
\geq f((ba)a \wedge S(a)) \geq f(a) \wedge 1 = f(a)$.

Now by using Lemma 1, $f \circ S = f$.

**Lemma 3:** Let $f$ and $g$ be any fuzzy two-sided ideals of a (2,2 regular $S$, then $f \circ g = f \cap g$.

**Proof:** Assume that $f$ and $g$ are any fuzzy two-sided ideals of $S$, then for every $a \in S$ there exists $y \in S$ such that $a \leq (a^2) \leq a^2$. Now by using (3), we have

$a \leq (a^2) \leq a^2$

for every fuzzy two-sided ideal $f$ of $S$. Thus $(a, xa) \in Aa$, since $Aa \neq \emptyset$, therefore
A subgroupoid of $S$ if and only if $f$.

**Lemma 5:** Every fuzzy two-sided ideal $f$ of a $(2,2)$ regular $S$ is idempotent.

Proof: Let $f$ be a fuzzy two-sided ideal of a $(2,2)$ regular $S$. For every $a \in S$ there exists $y \in S$ such that $a \leq (a'y) a^2$ and from Lemma 2, $((ba) a, a) \in A_a$.

Since $A_a \neq \emptyset$, therefore

$$
(f \circ g)(a) = \bigvee_{(ba, a) \in A_a} \{ f((ba)a) \wedge g(a) \} \\
\geq f(a) \wedge g(a) \\
= f(a).
$$

Thus by using Lemma 1, we get $f \circ g = f \cap g$.

**Lemma 4:** Every fuzzy two-sided ideal $f$ of a $(2,2)$ regular $S$ is idempotent.

Proof: Let $f$ be a fuzzy two-sided ideal of a $(2,2)$ regular $S$. For every $a \in S$ there exists $y \in S$ such that $a \leq (a'y) a^2$ and from Lemma 2, $((ba) a, a) \in A_a$.

Since $A_a \neq \emptyset$, therefore

$$
(f \circ g)(a) = \bigvee_{(ba, a) \in A_a} \{ f((ba)a) \wedge f(a) \} \\
\geq f(a) \wedge f(a) \\
= f(a).
$$

Now by using Lemma 1, $f \circ f = f$.

**Corollary:** Every fuzzy left ideal of a $(2,2)$ regular $S$ is idempotent.

**Theorem 2:** The set of fuzzy two-sided ideals of a $(2,2)$ regular $S$ forms a semilattice structure with identity $S$.

Proof: Let $F$ be the set of all fuzzy two-sided ideals of a $(2,2)$ regular $S$ and let $f, g, h$ be in $F$. Clearly $F$ is closed and by Lemma 4, we have $f \circ f = f$. Now by using Lemma 3, we get $f \cap g = g \circ f$. Therefore by using (3) and commutative law, we have

$$
(f \circ g) \circ h = (g \circ f) \circ h \\
= (h \circ f) \circ g \\
= (f \circ h) \circ g \\
= (g \circ h) \circ f \\
= f \circ (g \circ h).$$

It is easy to see from Lemma 2 that $S$ is an identity in $F$.

**Some fuzzy characterizations**

In this section, we have characterized fuzzy left (right) ideals, fuzzy two-sided ideals, fuzzy bi-ideals, fuzzy generalized bi-ideals, fuzzy $(1,2)$-ideals, fuzzy interior ideals and fuzzy quasi ideals of a $(2,2)$ regular ordered AG**-groupoid.

**Lemma 5:** A fuzzy subset $f$ of $S$ is a fuzzy AG**-subgroupoid of $S$ if and only if $f \circ f \subseteq f$.

Proof: It is simple.

**Theorem 3:** In a $(2,2)$ regular $S$, the following statements are equivalent.

(i) $f$ is a fuzzy bi-(generalized bi-) ideal.

(ii) $(f \circ S) \circ f = f$ and $f \circ f$.

Proof: (i) $\Rightarrow$ (ii): Assume that $f$ is a fuzzy bi-ideal of a $(2,2)$ regular $S$ and let $a \in S$, then there exists $y \in S$ such that $a \leq (a'y) a^2$. Now by using (4), (1) and (3), we have $a \leq (a'y) a^2$

$$
= (a'y)(aa) \\
= (aa)(ya^2) \\
\leq (x(x((aa)(ya^2)))a) \\
= (x((aa)a)x)a \\
= (aa)(xx)a x a \\
= (((aa)a)x)a x a \\
= (((aa)(aa)a)x)a x a \\
= (((aa)a)(aa)a)x a x a,
$$

where $ya^2 = x \in S$.

Thus $((a(x^2a))a)x, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$
((f \circ S) \circ f)(a) = \bigvee_{((a(x^2a))a)x, a) \in A_a} \{ f \circ S \{ ((a(x^2a))a)x \} \wedge f(a) \} \\
\geq f(a) \wedge f(a) \\
= f(a).
$$

Now by using (4), (1) and (3), we have $a \leq (a'y) a^2 = (aa)(ya^2)$

$$
= (aa)x = (xa)a \\
\leq (x((aa)(ya^2)))a \\
= (x((aa)a)x)a \\
= (aa)(aa)x a x a \\
= (((aa)a)x)a x a \\
= (((aa)(aa)a)x)a x a \\
= (((aa)a)(aa)a)x a x a \\
= (((aa)a)(aa)a)x a x a,
$$

where $ya^2 = x \in S$.

Thus $((a(x^2a))a)x, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$
(f \circ S) \circ f(a) = \bigvee_{((a(x^2a))a)x, a) \in A_a} \{ f \circ S \{ ((a(x^2a))a)x \} \wedge f(a) \} \\
= \bigvee_{((a(x^2a))a)x, a) \in A_a} \{ f \circ S \{ ((a(x^2a))a)x \} \wedge f(a) \}
$$

Thus $(f \circ S) \circ f = f$.

Now by using (3), (1), (5) and (4), we have $a \leq (a'y) a^2 = (aa)(ya^2)$

$$
= (aa)x = (xa)a \\
= (aa)(aa)x a x a \\
= (((aa)a)(aa)a)x a x a \\
= (((aa)a)(aa)a)x a x a \\
= (((aa)a)(aa)a)x a x a,
$$

where $ya^2 = x \in S$.

Thus $((a(x^2a))a)x, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$
(f \circ S) \circ f(a) = \bigvee_{((a(x^2a))a)x, a) \in A_a} \{ f \circ S \{ ((a(x^2a))a)x \} \wedge f(a) \} \\
= \bigvee_{((a(x^2a))a)x, a) \in A_a} \{ f \circ S \{ ((a(x^2a))a)x \} \wedge f(a) \}
$$

Thus $(f \circ S) \circ f = f$. 


Theorem 5: In a (2,2) regular S, the following statements are equivalent.

(i) \( f \) is a fuzzy interior ideal.

(ii) \( (S \circ f) \circ f = f \).

Proof: It is simple.

Theorem 6: A fuzzy subset \( f \) of a (2,2) regular S is a fuzzy right ideal of a (2,2) regular S if and only if it is a fuzzy left ideal of a (2,2) regular S.

Proof: Assume that \( f \) is a fuzzy right ideal of S. Since S is a (2,2) regular so for each \( f \) there exist \( x,y \in S \) such that \( a \leq (xa^2)y \). Now by using (1), we have

\[ f(ab) \geq f((xa^2)y)b \]
\[ = f(by)x(a^2) \]
\[ \geq f(by) \]
\[ \geq f(b). \]

Conversely, assume that \( f \) is a fuzzy left ideal of S. Now by using (1), we have

\[ f(ab) \geq f((xa^2)y)b \]
\[ = f(by)x(a^2) \]
\[ \geq f(x(a^2)) \]
\[ \geq f(aa) \]
\[ \geq f(a). \]
Proof: the following conditions are equivalent.

(i) \( f \) is a fuzzy two-sided ideal.
(ii) \( f \) is a fuzzy interior ideal.

Proof: (i) \( \Rightarrow \) (ii): Let \( f \) be any fuzzy two-sided ideal of a \((2,2)\) regular \( S \), then obviously \( f \) is a fuzzy interior ideal of a \((2,2)\) regular \( S \).

(iii) \( \Rightarrow \) (i): Let \( f \) be any fuzzy interior ideal of a \((2,2)\) regular \( S \) and let \( a, b \in S \). Since \( S \) is \((2,2)\) regular, so there exist \( x, y, u, v \in S \) such that \( a \leq (xa^2)y \) and \( b \leq (ub^2)v \). Thus by using (1), (3) and (2), we have

\[
\begin{align*}
\Rightarrow & f(ab) \geq f((xa^2)y)b \\
= & f((by)(x(aa))) \\
= & f((by)(a(xa))) \\
= & f((ba)(y(xa))) \\
\geq & f(a).
\end{align*}
\]

Also by using (3), (4) and (2), we have

\[
\begin{align*}
\Rightarrow & f(ab) \geq f((a(ub^2)v)y) \\
= & f((ub)(by)v) \\
= & f((ub)(by)(av)) \\
= & f((ub)(vu))(au) \\
= & f((ua)(vu)b) \\
\geq & f(b).
\end{align*}
\]

Hence \( f \) is a fuzzy two-sided ideal of a \((2,2)\) regular \( S \).

Theorem 8: A fuzzy subset \( f \) of a \((2,2)\) regular \( S \) is a fuzzy two-sided ideal if and only if it is a fuzzy quasi ideal.

Proof: It is simple.

Theorem 9: For a fuzzy subset \( f \) of a \((2,2)\) regular \( S \), the following conditions are equivalent.

(i) \( f \) is a fuzzy bi-ideal.
(ii) \( f \) is a fuzzy generalized bi-ideal.

Proof: (i) \( \Rightarrow \) (ii): Let \( f \) be any fuzzy bi-ideal of a \((2,2)\) regular \( S \), then obviously \( f \) is a fuzzy generalized bi-ideal of \( S \).

(ii) \( \Rightarrow \) (i): Let \( f \) be any fuzzy generalized bi-ideal of a \((2,2)\) regular \( S \), and \( a, b \in S \). Since \( S \) is \((2,2)\) regular so there exist \( x, y \in S \) such that \( a \leq (xa^2)y \). Now by using (3), (4), (2) and (3), we have

\[
\begin{align*}
\Rightarrow & f(ab) \geq f(((x(aa)y)b) \\
= & f(((x(aa)y)(vu))b) \\
= & f(((vu)(aa)x)b) \\
= & f(((vu)(va)x)b) \\
= & f(((vu)(va)u)b) \\
\geq & f(b).
\end{align*}
\]

Therefore \( f \) is a fuzzy two-sided ideal of a \((2,2)\) regular \( S \).

Theorem 10: For a fuzzy subset \( f \) of a \((2,2)\) regular \( S \), the following conditions are equivalent.

(i) \( f \) is a fuzzy two-sided.
(ii) \( f \) is a fuzzy bi-ideal.

Proof: (i) \( \Rightarrow \) (ii): Let \( f \) be any fuzzy two-sided ideal of a \((2,2)\) regular \( S \), then obviously \( f \) is a fuzzy bi-ideal of \( S \).

(ii) \( \Rightarrow \) (i): Let \( f \) be any fuzzy bi-ideal of a \((2,2)\) regular \( S \). Since \( S \) is a \((2,2)\) regular so for each \( a, b \in S \) there exist \( x, y, u, v \in S \) such that \( a \leq (xa^2)y \) and \( b \leq (ub^2)v \). Now by using (1), (4), (2) and (3), we have

\[
\begin{align*}
\Rightarrow & f(ab) \geq f(((xa^2)y)b) \\
= & f((by)(xa^2)) \\
= & f((a^2x)(by)) \\
= f(((aa)(xy))(a^2x)) \\
= f(((a(ub^2)v)y)b) \\
\geq & f(a).
\end{align*}
\]

Now by using (3), (4) and (1), we have

\[
\begin{align*}
\Rightarrow & f(ab) \geq f((a(ub^2)v)y) \\
= & f((a(ub^2)v))y \\
= f(((aa)(v(ub^2)v))y) \\
\geq & f((vu)(va)u)b) \\
= f((va)(b^2u)) \\
= f((vu)(va)u)b) \\
\geq & f(b).
\end{align*}
\]

Therefore \( f \) is a fuzzy two-sided ideal of a \((2,2)\) regular \( S \).

Theorem 11: For a fuzzy subset \( f \) of a \((2,2)\) regular \( S \), the following conditions are equivalent.

(i) \( f \) is a fuzzy left ideal of a \((2,2)\) regular \( S \).
(ii) \( f \) is a fuzzy right ideal of a \((2,2)\) regular \( S \).
(iii) \( f \) is a fuzzy \((1,2)\)-ideal of a \((2,2)\) regular \( S \).
(iv) \( f \) is a fuzzy two-sided ideal of a \((2,2)\) regular \( S \).
(v) \( f \) is a fuzzy bi-ideal of a \((2,2)\) regular \( S \).
(vi) \( f \) is a fuzzy generalized bi-ideal of a \((2,2)\) regular \( S \).
(vii) \( f \) is a fuzzy interior ideal of a \((2,2)\) regular \( S \).
(viii) \( f \) is a fuzzy quasi ideal of a \((2,2)\) regular \( S \).
(ix) \( S \circ f = f \circ S = S \).
Proof: (i) \(\Rightarrow\) (ix): It follows from Lemma 2.
(ix) \(\Rightarrow\) (viii) is obvious and (viii) \(\Rightarrow\) (vii) is simple.
(vii) \(\Rightarrow\) (vi): It follows from Theorems 6, 10 and 9.
(vi) \(\Rightarrow\) (v): It follows from Theorem 9.
(v) \(\Rightarrow\) (iv): It follows from Theorem 10.
(iv) \(\Rightarrow\) (iii): It follows from Theorem 7.
(iii) \(\Rightarrow\) (ii) and (ii) \(\Rightarrow\) (i) can be followed from Theorem 7 and Lemma 6.

REFERENCES