The Maximal Subgroups Of The Orthogonal Group $\text{PO}^+(10, 2)$

Rauhi I. Elkhatif
Dept. of Mathematics, Faculty of Applied Science, Thamar University, Yemen
E-mail: Rauhie@yahoo.com

ABSTRACT

The main result of this work is called "the main theorem" which is a list of maximal subgroups of the orthogonal group $\text{PO}^+(10, 2)$ which has been proved by using Aschbacher’s Theorem ([1]). So, this work is divided into two main parts: Part (1): In this part, we will find the maximal subgroups in the classes $C_1 - C_8$ of Aschbacher’s Theorem ([1]). Part (2): In this part, we will find the maximal subgroups in the class $C_9$ of Aschbacher’s Theorem ([1]), which are the maximal primitive subgroups $H$ of $G$ that have the property that the minimal normal subgroup $M$ of $H$ is not abelian group and simple, thus, we divided this part into two cases:

Case (1): $M$ is generated by transvections: In this case, we will use result of Kantor ([2]).

Case (2): $M$ is a finite primitive subgroup of rank three: In this case, we will use the classification of Kantor and Liebler ([3]).

Mathematics Subject Classification: 20B05; 20G40, 20H30, 20E28.

Keywords: Finite groups; linear groups, matrix groups, maximal subgroups.

1. INTRODUCTION

The purpose of this paper is to prove the following main theorem:

Theorem 1.1: Let $G = \text{PO}^+(10, 2)$. If $H$ is a maximal subgroup of $G$, then $H$ isomorphic to one of the following subgroups:

1. $H_1 = 2^8 \langle \text{PGL}(1, 2) \times \text{PO}^+(8, 2) \rangle$.
2. $H_2 = 2^{14} \langle \text{PGL}(2, 2) \times \text{PO}^+(6, 2) \rangle$.
3. $H_3 = 2^{14} \langle \text{PGL}(3, 2) \times \text{PO}^+(4, 2) \rangle$.
4. $H_4 = 2^{14} \langle \text{PGL}(4, 2) \times \text{PO}^+(2, 2) \rangle$.
5. $H_5 = 2^{10} \langle \text{PGL}(5, 2) \rangle$.
6. $H_6 = \text{PSO}^+(2, 2) \times \text{PO}^+(8, 2)$.
7. $H_7 = \text{PO}^+(4, 2) \times \text{PO}^+(6, 2)$.
8. $H_8 = \text{PO}^+(4, 2) \times \text{PO}^+(6, 2)$.
9. $H_9 = \text{PSp}(8, 2)$.
10. $H_{10} = \text{PO}^+ (2, 2^5).5$.

We will prove this theorem by Aschbacher’s theorem (Result 2.9) [see [1]], but before starting the proof of this theorem, we will introduce some background:

Definition 1.2: Let $V$ be a finite n-dimensional vector space over a field $F$. A map $f: V \rightarrow V$ is called semi-linear with respect to an automorphism $\theta$ of $F$ if:

(i) $f(x + y) = f(x) + f(y)$, $\forall x, y \in V$;
(ii) $f(ax) = a^\theta f(x)$, $\forall a \in F$, $x \in V$.

The maps $f$ with $\theta = 1$ are called Linear. The set of all invertible linear transformations of $V$ into itself form a group called the general linear group and is denoted by $\text{GL}(n, F)$. If $F$ is a finite field with $q$ elements then $\text{GL}(n, F)$ is denoted by $\text{GL}(n, q)$. The centre $Z$ of $\text{GL}(n, q)$ is the set of all non-singular scalar matrices, hence, we may form the factor group $\text{GL}(n, q) / Z$ called The projective general linear group which is denoted by $\text{PGL}(n, q)$. $\text{GL}(n, q)$ has a normal subgroup $\text{SL}(n, q)$, consisting of all matrices of determinant 1 called the special linear group. The image of $\text{SL}(n, q)$ under the mapping $\phi: \text{GL}(n,q) \rightarrow \text{PGL}(n, q)$ is the projective special linear group and is denoted by $\text{PSL}(n, q)$. $\text{PSL}(n, q)$ is simple, except for $\text{PSL}(2, 2)$ and $\text{PSL}(2, 3)$. $\Gamma \text{L}(n, q)$ is the group of all invertible semi-linear transformations of $V$, $\text{PGL}(n, q) \cong \Gamma \text{L}(n, q) / Z$. In the matrix form, the general linear group $\text{GL}(n, F)$ can be defined by the set of all invertible matrices, i.e., $\{g \in M_n(F): \det(g) \neq 0\}$ and the
special linear group $SL(n, F) = \{ g \in GL(n, F) : \det(g) = 1 \}$.

**Definition 1.3:** Let $V$ be a finite $n$-dimensional vector space over a finite field $F$ with $q$ elements and $\theta$ be an automorphism of $F = GF(q)$. A *sesquilinear form* $B$ on $V$ is a mapping $B : V \times V \rightarrow F$ such that:

1. $B(x_1 + x_2, y_1 + y_2) = B(x_1, y_1) + B(x_2, y_1) + B(x_1, y_2) + B(x_2, y_2)$, $\forall \ x_1, x_2, y_1, y_2 \in V$.

2. $B(ax, by) = a b^\theta B(x, y)$, $\forall \ a, b \in F$ and $\theta \in Aut(F)$ is of order 1 or 2.

$B$ is called *non-degenerate* if $B(x, y) = 0, \forall \ y \in V \Rightarrow x = 0$. If $\theta = 1$, then $B$ is called *Bilinear form*.

**Definition 1.4:** Let $B$ be a non-degenerate Bilinear form on an $n$-dimensional vector space $V$ over a finite field $F_q$ which has the property that $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$ for all $x, y \in V$. Then the subgroup of $GL(n, q)$ which preserves the form $B(x, x) = 0, \forall \ x \in V$; is called *symplectic group* and is denoted by $Sp(n, q)$. The *projective symplectic group* $GSp(2m, q)$ is the group induced on the set of points of $PG(2m-1, q)$ by $Sp(2m, q)$, thus, $GSp(2m, q) \cong Sp(2m, q) / (Sp(2m, q) \cap SL(n, F))$. The *general symplectic group* $GU(n, q) \cong SU(n, q) / SU(n, q) \cap GL(n, q)$.

**Definition 1.6:** Let $V$ be a vector space and $v \in V$. A *reflection* $\sigma_v$ of $V$ is a map takes $v \rightarrow -v$ for every $v$ in $V$ and fixes every vector orthogonal to $v$, this map is given by $\sigma_v : x \rightarrow x - \frac{2B(x, v)}{B(v, v)}$, $x \in V$. Any reflection has determinant $-1$.

**Definition 1.7:** Let $B$ be a non-degenerate Bilinear form on an $n$-dimensional vector space $V$ over a finite field $F_q$, which has the property that $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$ for all $x, y \in V$. Then the subgroup of $GL(n, q)$ which preserves the form $B(x, y) = B(y, x), \forall x, y \in V$ is called *orthogonal group* and is denoted by $O(n, q)$. The *special orthogonal group* $SO(n, q)$ consists of all elements of the orthogonal group which have determinant 1 (that is, $SO(n, q) = O(n, q) \cap SL(n, q)$). A *quadratic form* $Q$ on $V$ is the mapping $Q(x) = \frac{1}{2}B(x, x)$. A subspace $U$ of $V$ is *isotropic* if $B(x, y) = 0, \forall x, y \in U$, all maximal isotropic subspaces have the same dimension $\mu$ is called a *Witt index* of $Q$. In the odd dimension $n = 2m+1$, there is a unique form with Witt index $\mu = m$ and the corresponding orthogonal group is denoted by $O(n, q)$. In the even dimension $n = 2m$, there are two forms, the first one has Witt index $\mu = m$ and the corresponding orthogonal group is denoted by $O^+(n, q)$. The second has Witt index $\mu = m-1$ and the corresponding orthogonal group is denoted by $O^-(n, q)$. In the matrix case, the *orthogonal group* $O(n, q) = \{ g \in GL(n, q) : B^g = I_n \}$ or $O(n, q) = \{ g \in GL(n, q) : g^T J g = I_n \}$, where $J$ is the diagonal matrix with entries $1, 1, \ldots, 1$ for non square $\lambda$. These amounts are the same if $n$ is odd, but when $n$ is even the two possibilities of orthogonal group give us two different results depending on $n$ and $q$, these groups are written as $O^\varepsilon(n, q)$ with $\varepsilon = \pm 1$. Above $O(n, q)$ there is the semi linear orthogonal group $GO(n, q)$ and the general orthogonal group $GO^0(n, q) = \Gamma O(n, q) \cap GL(n, q)$. Below $O(n, q)$ there is the special orthogonal group $O(n, q) \cap SL(n, q) = \{ g \in O(n, q) : \det(g) = 1 \}$. Further, the projective orthogonal group $PO(n, q) =...
O(n, q) / (O(n, q) ∩ Z), where Z consists of the scalar matrices, also, the projective special orthogonal group
PSO(n, q) = SO(n, q) / (SO(n, q) ∩ Z). For q and n are
even, we have O(n, q) = SO(n, q) = PO(n, q) =
PSO(n, q). The commutator (the second derived) subgroup of O(n, q) is Ω(n, q) and the corresponding factor group
PΩ(n, q) = Ω(n, q) / (Ω(n, q) ∩ Z) ≅ PΩ(n, q) is simple for
all n ≥ 3 except PΩ³(4, q), PΩ(3, 3) and PΩ(2, q). Always, Ω(n, q) ≤ SO(n, q) and if q even, then Ω(n, q) =
SO(n, q) except when n = 4 and q = 2. In the even
dimensions n = 2m, if q^n ≠ ε mod 4 then SO²(2m, q) =
2 × Ω²(2m, q) and PSO²(2m, q) = PΩ²(2m, q). If q is even and n is odd dimensions 2m + 1, the orthogonal group
is the same as the symplectic group of dimension 2m.
Thus, O(2m + 1, q) ≅ Sp(2m, q) for q even. In even
dimension, the orthogonal group is a subgroup of the
symplectic group. In general, Sp(2m − 2, q) ≅
Ω²(2m − 1, q) is a maximal subgroup of Ω²(2m, q). If
the field q of characteristic 2, then the orthogonal group
is generated by orthogonal transvections. If q is odd
characteristic, then there are no orthogonal transvections
but in this case the orthogonal group is generated by
reflections.

Finally, the orders of the orthogonal groups are given by:

1. |O(2m + 1, q)| = |O(2m + 1, q)| = |PΩ(2m + 1, q)| = 12, q − 1m2i = 1m2i − 1;

2. |O²(2m, q)| = |O²(2m, q)| = 1
4, 2q⁴m + 1(q²m + 1)(q²m + 1) Πi=1, q²m + 1 → 1;

3. |Ω²(2m, q)| = 1
2(q²m + 1)(q²m + 1) Πi=1, q²m + 1 → 1.

Note 1.8: Isomorphisms of the classical groups:

- PSL(2, 2) ≅ S₃ (not simple).
- PGL(2, 3) ≅ S₄ (not simple).
- PGL(2, 5) ≅ S₅ (not simple).
- PSL(2, 3) ≅ A₄ (not simple).
- PSp(4, 2) ≅ S₆ (not simple).
- PGL(2, 4) ≅ PSL(2, 4) ≅ PSL(2, 5) ≅ A₅.
- PSL(2, 7) ≅ PSL(3, 2).

- PSL(2, 9) ≅ A₅.
- PSL(4, 2) ≅ A₄.
- PSp(2, q) ≅ PSL(2, q).
- PSU(4, 2) ≅ PSp(4, 3).
- GL(2, q) ≅ SL(2, q) ≅ SU(2, q²).
- O(2m+1, q) ≅ Sp(2m, q) for q even.
- O²(2, q) ≅ D₁(2q-1).
- O²(2, q) ≅ D₂(2q-1).
- PSO(3, q) ≅ PGL(2, q).
- PSO³(4, q) ≅ (PSL(2, q) × PSL(2, q)).
- PSL(2, q) ≅ PSL(2, q)².
- PSL(5, q) ≅ PSp(4, q).
- PSL(6, q) ≅ PSL(2, q)².
- PSL(3, q) ≅ PSL(2, q).
- PSL(2, q)₂ ≅ PSL(2, q)².
- PSL(6, q) ≅ PSL(4, q).
- PSL(6, q) ≅ PSL(4, q).
- PSL(6, q) ≅ PSL(4, q).

Through this article, G will denote PΩ³(10, 2) unless
otherwise stated, G is a simple group of order
2²0.3⁵.5².17.13.31 = 2349929548800 and G acts
primitively on the points of the projective space PG(9, 2)
which is a rank 3 permutation group on PG(9, 2) (see [4],
[5], [6], [7], [8]).

2. ASCHBACHER’S THEOREM

A classification of the maximal subgroups of
GL(n, q) by Aschbacher’s theorem (see [1]), is a very
strong tool in the finite groups for finding the maximal
subgroups of finite linear groups. There are many good
works in finite groups which simplify this theorem, see for
example ([9]). But before giving a brief description of this
theorem, we will give the following definitions:

Definition 2.1: A split extension (a semidirect product )
A:B is a group G with a normal subgroup A and a
subgroup B such that G = AB and A ∩ B = 1. A non-split
extension A.B is a group G with a normal subgroup A
and G/A ≅ B, but with no subgroup B satisfying G = AB
and A ∩ B = 1. A group G = A ∘ B is a central product of
its subgroups A and B if G = AB and [A, B], the commutator
of A and B = 1, in this case A and B are normal
subgroups of G and A ∩ B ≤ Z(G). If A ∩ B = {1}, then A ∘ B = AB.
Definition 2.2: Let V be a vector space of dimensional n over a finite field q, a subgroup H of GL(n, q) is called reducible if it stabilizes a proper nontrivial subspace of V. If H is not reducible, then it is called irreducible. If H is irreducible for all field extension F of F_q, then H is absolutely irreducible. An irreducible subgroup H of GL(n, q) is called imprimitive if there are subspaces V_1, V_2, ..., V_k, k ≥ 2, of V such that V = V_1 ⊕ ... ⊕ V_k and H permutes the elements of the set {V_1, V_2, ..., V_k} among themselves. When H is not imprimitive then it is called primitive.

Definition 2.3: A group H ≤ GL(n, q) is a superfield group of degree s if for some s divides n with s > 1, the group H may be embedded in GL(n/s, q^s).

Definition 2.4: If the group H ≤ GL(n, q) preserves a decomposition V = V_i ⊕ V_j, with dim(V_i) ≠ dim(V_j), then H is a tensor product group.

Definition 2.5: Suppose that n = r^m and m > 1. If the group H ≤ GL(n, q) preserves a decomposition V = V_i ⊕ ... ⊕ V_m with dim(V_i) = r for 1 ≤ i ≤ m, then H is a tensor induced group.

Definition 2.6: A group H ≤ GL(n, q) is a subfield group if there exists a subfield F_q’ ⊂ F_q such that H can be embedded in GL(n, q’), where Z is the centre group of H.

Definition 2.7: A p-group H is called a special group if |Z(H)| = p. Thus the automorphism group of T.

A classification of the maximal subgroups of GL(n, q) by Aschbacher’s theorem [1], can be summarized as follows:

Result 2.9. (Aschbacher’s theorem):

Let H be a subgroup of GL(n, q), q = p^s with the centre Z and let V be the underlying n-dimensional vector space over a field q. If H is a maximal subgroup of GL(n, q), then one of the following holds:

C_1: H is a reducible group.

C_2: H is an imprimitive group.

C_3: H is a superfield group.

C_4: H is a tensor product group.

C_5: H is a subfield group.

C_6: H normalizes an irreducible extraspecial or symplectic-type group.

C_7: H is a tensor induced group.

C_8: H normalizes a classical group in its natural representation.

H is absolutely irreducible and H/(H∩Z) is almost simple.

3. Classes C_1 − C_8 of Result 2.9

In this section, we will find the maximal subgroups in the classes C_1 − C_8 of Result 2.9:

Lemma 3.1: There are nine reducible maximal subgroups of C_1 in G which are:

1. H_1 = 2^4:(PGL(1, 2)×PΩ^+(8, 2)).
2. H_2 = 2^{10}:(PGL(2, 2)×PΩ^+(6, 2)).
3. H_3 = 2^{12}:(PGL(3, 2)×PΩ^+(4, 2)).
4. H_4 = 2^{14}:(PGL(4, 2)×PΩ^+(2, 2)).
5. H_5 = 2^{10}:PGL(5, 2).
6. H_6 = PΩ^+(2, 2)×PΩ^+(6, 2).
7. H_7 = PΩ^+(4, 2)×PΩ^+(6, 2).
8. H_8 = PΩ^+(4, 2)×PΩ^+(6, 2).
9. H_9 = PSp(8, 2).

Proof: If H be a reducible subgroup of the orthogonal group O^+(2n, q), then we have three cases:

Case 1: Let W be an invariant subspace of H, k = dim (W), 1 ≤ k ≤ n and G_k = G(W) denote the subgroup of O^+(2n, q) containing all elements fixing W as a whole and H ⊆ G_k with a suitable choice of a basis, G_k consists of all matrices of the form

\[
\begin{pmatrix}
A & D & E \\
B & F & 0 \\
0 & 0 & A
\end{pmatrix}
\]

where A is a p-group of upper triangular matrix of identity diagonal of order q^{k(k−1)} / 2, D and F are two elementary abelian groups of order q^{n-2k} / 2, E ∈ GL(k, q) and B ∈ O^+(2n-2k, q) such that B^t B = I_n or B^t J B = J, and where J is the diagonal matrix with entries 1, 1, ..., λ for non square λ ∈ q^×. Thus the maximal parabolic subgroups are the stabilizers of totally
isotropic subspaces \(<e_1, e_2, \ldots, e_k>\) is isomorphic to a group of the form \(\Omega_{k+1}(2n)\times GL(k, q)\). Thus, we have the following reducible maximal subgroups of \(\Omega^{\epsilon}(10, 2)\):

1. If \(k = 1\), then we get a group, stabilizing a point isomorphic to a group of the form \(H_1 = 2^2: (PGL(1, 2)\times \Omega^\epsilon(2n-2, q))\).
2. If \(k = 2\), then we get a group, stabilizing a line is isomorphic to a group of the form \(H_2 = 2^3: (PGL(2, 2)\times \Omega^\epsilon(6, 2))\).
3. If \(k = 3\), then we get a group, stabilizing a plane is isomorphic to a group of the form \(H_3 = 2^5: (PGL(3, 2)\times \Omega^\epsilon(4, 2))\).
4. If \(k = 4\), then we get a group, stabilizing a 3-space is isomorphic to a group of the form \(H_4 = 2^8: (PGL(4, 2)\times \Omega^\epsilon(2, 2))\).
5. If \(k = 5\), then we get a group, stabilizing a 4-space is isomorphic to a group of the form \(H_5 = 2^{10}: PGL(5, 2)\).

**Case 2:** If \(H\) is the maximal reducible subgroup of the orthogonal group \(O^\epsilon(2n, q)\) which stabilizers of non-singular subspaces of dimension 2k, then for \(n = k + m\) and \(1 \leq k < m\), we have \(H = O^\epsilon(2k, q)\times O^\epsilon(2m-2k, q)\), also since the quadratic form \(x^2 + xy^2\) is equivalent to \(x^2 + y^2\) for the non square \(x^2 + y^2\), then \(H = O(2k, q)\times O(2m-2k, q)\). In this case, \(H\) consists of all matrices of the form

\[
\begin{pmatrix}
A & 0 & B \\
0 & C & 0 \\
D & 0 & E
\end{pmatrix}
\]

where \((A, B) \in \Omega^\epsilon(2k, q)\) . Thus, we have the following reducible subgroups of \(\Omega^{\epsilon}(10, 2)\):

6. If \(k = 1\) and \(m = 4\), then we get the group \(H_6 = \Omega^\epsilon(2, 2)\times \Omega^\epsilon(8, 2)\).
7. If \(k = 1\) and \(m = 4\), then we get the group \(H_7 = \Omega^\epsilon(2, 2)\times \Omega^\epsilon(8, 2)\).
8. If \(k = 2\) and \(m = 3\), then we get the group \(H_8 = \Omega^\epsilon(4, 2)\times \Omega^\epsilon(6, 2)\).
9. If \(k = 2\) and \(m = 3\), then we get the group \(H_9 = \Omega^\epsilon(4, 2)\times \Omega^\epsilon(6, 2)\).

But \(H_6 < H_1\), then \(H_6\) is not maximal subgroup of \(G\).

**Case 3:** Since, \(O(2n-1, q) = Sp(2n-2, q)\) for \(q\) even, and \(O(2n-1, q)\) is a maximal reducible subgroup of \(O^\epsilon(2n, q)\), then for \(q\) even, \(Sp(2n-2, q)\) is a maximal subgroup of \(O^\epsilon(2n, q)\). Consequently, \(PSp(8, 2)\) is a maximal subgroup of \(\Omega^{\epsilon}(10, 2)\). Which prove the points (1), (2), (3), (4), (5), (6), (7), (8) and (9) of the main theorem 1.1.

**Lemma 3.2:** There is no imprimitive group of \(C_2\) in \(G\).

*Proof:* If \(H\) is imprimitive of the orthogonal group \(O^\epsilon(2n, q)\), then \(H\) preserves a decomposition of \(V\) as a direct sum \(V = V_1 \oplus \cdots \oplus V_t\) \(t \geq 2\), into subspaces of \(V\), each of dimension \(m = n/t\), which are permuted transitively by \(H\), thus \(H\) is isomorphic to \(O^\epsilon(2m, q):S_t\) with \(0 < m < n = mt\), \(t \geq 2\). Consequently, there is one imprimitive group of \(C_2\) in \(\Omega^{\epsilon}(10, 2)\) which is \(H_{11} = \Omega^\epsilon(2, 2)\times S_5\), a group preserving five mutually lines of projective plane \(PG(9, 2)\) and \(H_1\) interchanges them. But \(\Omega^\epsilon(1, q) \cong \Omega^\epsilon(2, 2)\) and \(H_1\) interchanges them. Thus \(H_{11} < \Omega^\epsilon(4n, 2)\) where \(n \geq 1\), thus \(S_t < \Omega^\epsilon(4, 2)\). Thus \(H_{11}\) is not a maximal subgroups of \(\Omega^{\epsilon}(10, 2)\).

**Lemma 3.3:** There is one semilinear group of \(C_3\) in \(G\) which is \(H_{12} = \Omega^\epsilon(2, 2^5)\).

*Proof:* Let \(H\) is (superfield group) a semilinear groups of \(O^\epsilon(2n, q)\) over extension field \(F_r\) of \(GF(q)\) of prime degree \(r > 1\) where \(r\) prime number divide \(n\). Thus \(V\) is an \(F_r\)-vector space in a natural way, so there is an \(F\)-vector space isomorphism between \(2n\)-dimensional vector space over \(F\) and the \((m - 1)\)-dimensional vector space over \(F\), where \(m = n/r\), thus \(H\) embeds in \(O^\epsilon(2m, q)^r\). Consequently, there is one \(S_{11}\) group in \(\Omega^\epsilon(10, 2)\) which is \(H_{12} = \Omega^\epsilon(2, 2^5)\). This proves the point (10) of the main theorem 1.1.

**Lemma 3.4:** There is no tensor product group in \(G\).

*Proof:* If \(H\) is a tensor product group of \(O^\epsilon(2n, q)\), then \(H\) preserves a decomposition of \(V\) as a tensor product \(V_1 \otimes V_2\), where \(dim(V_1) \neq dim(V_2)\) of spaces of dimensions \(2k\) and \(2m\) over \(GF(q)\) and \(2n = 4km\), \(k \neq m\). So, \(H\) stabilizes the tensor product decomposition \(F^{2k} \otimes F^{2m}\). Thus, \(H\) is a subgroup of the central product of \(O^\epsilon(2k, q) \times O^\epsilon(2m, q)\). Consequently, there is no tensor product group in \(\Omega^\epsilon(10, 2)\), since \(5\) is a prime number.

**Lemma 3.5:** There is no subfield group of \(C_3\) in \(G\).

*Proof:* If \(H\) is a subfield group of the orthogonal group \(O^\epsilon(2n, q)\) and \(q = p^k\), then \(H\) is the orthogonal group over subfield of \(GF(q)\) of prime index. Thus \(H\) can be embedded in \(O^\epsilon(2n, p^k)\), where \(k\) prime number divides \(f\). Consequently, since \(2\) is a prime number, then there is no subfield group of \(C_3\) in \(G\).
Lemma 3.6: There are no $C_{6}$ groups in $G$.

Proof: For the dimension $2n = r = m$ and $r$ is prime of the orthogonal group $O^r(2n, q)$. If $r$ is odd prime and $r$ divides $q - 1$, then $H = 2^{2m+1}$. $\Omega(2m, 2)$ normalizes an extraspecial $r$-group which fixes the symplectic form. Otherwise if $r = 2$ and $4$ divides $q - 1$, then $H = 2^{2m}O^r(2m, 2)$ normalizes an extraspecial 2-group which fixes the symplectic form (see [5]). Consequently, there are no $C_{6}$ groups in $G$. Since $10$ is not a prime power.

Lemma 3.7: There is no tensor induced group of $C_{7}$ in $G$.

Proof: If $H$ is a tensor induced of the orthogonal group $O^r(2n, q)$, then $H$ preserves a decomposition of $V$ as $V_1 \oplus V_2 \oplus \ldots \oplus V_n$, where $V_i$ are isomorphic, each $V_i$ has dimension $2m$, $\dim V = 2n = (2m)^i$, and the set of $V_i$ is permuted by $H$, so $H$ stabilizes the tensor product decomposition $F^{2m} \otimes F^{2m} \otimes \ldots \otimes F^{2m}$, where $F = F_q$. Thus, $H/ \leq PO^r(2m, q)$, $S_n$. Consequently, there is no tensor induced group in $P\Omega(10, 2)$, since $n = 10$ is not prime power.

Lemma 3.8: There are no $C_{6}$ groups in $G$.

Proof: The groups in this class are stabilizers of forms, this means $H$ is the normalizer of one classical groups $PSL(2n, q)$, $PSp(2n, q)$ or $PSU(2n, q)$ as a subgroup of $P\Omega(2n, q)$. Let $H$ be a proper irreducible subgroup of $G$ with a minimal normal subgroup $M$ of $H$ which is not abelian and simple. We will prove this Result 2.9: Let $H$ be a proper irreducible subgroup of $G$ with a minimal normal subgroup $M$ of $H$ which is not abelian and simple.

4.1 Primitive subgroups $H$ of $G$ which have the property that a minimal normal subgroup $M$ of $H$ is not abelian is generated by transvections

Definition 4.1.1: An element $T \in \text{GL}(n, q)$ is called a transvection if $T$ satisfies $\text{rank}(T - I_n) = 1$ and $(T - I_n)^2 = 0$. The collineation of projective space induced by a transvection is called elation. The axis of the transvection is the hyperplane $\text{Ker}(T - I_n)$; this subspace is fixed elementwise by $T$. Dually, the centre of $T$ is the image of $(T - I_n)$.

To find the primitive subgroups $H$ of $G$ which have the property that a minimal normal subgroup $M$ of $H$ is not abelian and is generated by transvections, we will use the following result of Kantor [2]:

Result 4.1.2: Let $H$ be a proper irreducible subgroup of $\Omega^r(m, s)$ generated by transvections. Then $H$ is one of:

1. $H = \Omega^r(2n, q) \leq \Omega^r(2n, q^3)$;
2. $H = SU(2n, q) \leq \Omega^r(4n, q)$.
3. \( H = SU(2n+1, q) < \Omega(4n+2, q) \);
4. \( H/Z(G) = P\Omega(7, q), |Z(G)| = (2, q - 1), H < \Omega^\prime(8, q) \);
5. \( H = \Omega^\prime(4, q) < \Omega(5, q), q \text{ even} \);
6. \( H = HJ \text{ “Hall Janko group”} < \Omega(7, 4) \);
7. \( H = GU(6, 2) < \Omega^\prime(12, 2) \);
8. \( H = A:S_n < \Omega^\prime(2n, q), q \text{ even}, + \text{ for } n \text{ even}, - \text{ for } n \text{ odd}, A \text{ the direct product of } n \text{ cyclic group of order } a|q \pm 1; \)
9. \( H = (A:E):S_n < \Omega^\prime(2n, q), q \text{ even}, + \text{ for } n \text{ even}, - \text{ for } n \text{ odd}, A \text{ the direct product of } n \text{ cyclic group of order } a|q \pm 1, E \text{ an elementary abelian } 2\text{-group of order } 2^{n\pm 1}; \)
10. \( H = SL(n, q) < \Omega^\prime(2n, q) \);
11. \( H = Sp(n, q) < \Omega^\prime(2n, q^2) \);
12. \( H = SU(n, q) < \Omega^\prime(2n, q^2) \);
13. \( H = O^\prime(n, q) < \Omega^\prime(2n, q^2), q \text{ even} \);
14. \( H = S_{4n+2} < \Omega(4n+1, 2), n\geq 2; \)
15. \( H = S_{4n+1} < \Omega(4n, 2), n\geq 2; \)
16. \( H = S_{4n+4} < \Omega(4n+2, 2), n\geq 2; \)
17. \( H = 3, A_6 < \Omega^\prime(6, 4) \);
18. \( H = SL(2, 5) < SL(2, 9) < \Omega^\prime(4, 9) \);
19. \( H = GU(6, 2) < \Omega^\prime(12, 4) \);
20. \( H = SU(4, 2) < GL(5, 4) < \Omega^\prime(10, 4) \);
21. \( H = A:S_n < SL(n, 2^i) < \Omega^\prime(2n, 2^i), A \text{ the direct product of } n \text{ cyclic group of order } a|q \pm 1, i > 1; \)
22. \( H = \Omega(2n-1, q) < \Omega^\prime(2n, q), q \text{ even}; \)
23. \( H = O^\prime(4, q) < \Omega(5, q) < \Omega^\prime(6, q), q \text{ even} \);
24. \( H = HJ \text{ “Hall Janko group”} < \Omega^\prime(8, 4) \);
25. \( H = SU(4, 2) < \Omega^\prime(10, 2) \);
26. \( H = SU(4, 2) < \Omega^\prime(10, 4) \);

In the following corollary, we will find the primitive subgroups of the orthogonal group \( P\Omega^\prime(10, 2) \) which is generated by transvections:

**Corollary 4.1.3:** If \( M \) is a non abelian simple group and contains some transvections, then \( M \) is isomorphic to one of the following groups:

(i) \( PSL(5, 2) \);
(ii) \( PSp(5, 2) \);
(iii) \( PO^\prime(5, 2) \);
(iv) \( PO^\prime(5, 2) \);

**Proof:** We will discuss the different possibilities of Result (4.1.2), so, \( M \) is isomorphic to one of the following groups:

1. If \( \Omega(2n, q) < \Omega^\prime(2n, q^2) = G \), then \( q^2 = 2 \) which is refused because \( q \) is a prime integer;
2. If \( SU(2n, q) < \Omega^\prime(4n, q) = G \), then \( 4n = 10 \) which is refused because \( n \) is a positive integer;
3. \( SU(2n+1, q) < \Omega(4n+2, q) = G \), it is not our case, since our case \( G = \Omega^\prime(10, 2) \);
4. \( H/Z(G) = P\Omega(7, q), |Z(G)| = (2, q - 1), H < \Omega^\prime(8, q) = G \), it is not our case, since our case \( G = \Omega^\prime(10, 2) \);
5. \( O^\prime(4, q) < \Omega(5, q) = G, q \text{ even} \), it is not our case, since our case \( G = \Omega^\prime(10, 2) \);
6. \( HJ \text{ “Hall Janko group”} < \Omega(7, 4) = G \), it is not our case, since our case \( G = \Omega^\prime(10, 2) \);
7. \( GU(6, 2) < \Omega^\prime(12, 2) = G \), it is not our case, since our case \( G = \Omega^\prime(10, 2) \);
8. \( A:S_n < \Omega^\prime(2n, q) = G, q \text{ even} \), for \( n \) even, - for \( n \) odd, \( A \) the direct product of \( n \) cyclic group of order \( a|q \pm 1 \), it is not our case, since if \( n = 5 \) odd, then \( G = \Omega(10, q) \) but our case \( G = \Omega^\prime(10, 2) \);
9. \( (A:E):S_n < \Omega^\prime(2n, q) = G, q \text{ even} \), for \( n \) even, - for \( n \) odd, \( A \) the direct product of \( n \) cyclic group of order \( a|q \pm 1 \), \( E \) an elementary abelian \( 2\text{-group of order } 2^{n\pm 1} \), it is not our case, since if \( n = 5 \) odd, then \( G = \Omega(10, q) \) but our case \( G = \Omega^\prime(10, 2) \);
10. If $SL(n, q) < \Omega^2(2n, q) = G$, then $SL(5, 2) < \Omega(10, 2)$, Consequently $PSL(5, 2) < P\Omega(10, 2)$;

11. If $Sp(n, q) < \Omega^2(2n, q) = G$, then $Sp(5, 2) < \Omega(10, 2)$, Consequently $PSp(5, 2) < P\Omega(10, 2)$;

12. If $SU(n, q) < \Omega^2(2n, q^2) = G$, then $q^2=2$ which is refused because $q$ is a prime integer;

13. If $O^+(n, q) < \Omega^2(2n, q) = G$, $q$ even, then $O^+(5, 2) < \Omega(10, 2)$, Consequently $PO^+(5, 2) < P\Omega(10, 2)$;

14. If $H = S_{4n+2} < \Omega(4n+1, 2), n\geq 2$, then $4n+1=10$ which is refused because $n$ is a positive integer;

15. If $S_{4n+1} < \Omega^2(4n, 2), n\geq 2$, then $4n=10$ which is refused because $n$ is a positive integer;

16. If $S_{4n+4} < \Omega^2(4n+2, 2) = G, 4n+2 = 10, then n=2$, and the irreducible 2-modular characters for $S_{12}$ by GAP:

```gap
gap> g:=GO(9,2);
```

CharacterTable( "4.2^4.3S5" )

```gap
gap> k2:=CharacterTable(c,2);
```

BrauerTable( "4.2^4.3S5", 2 )

```gap
gap> OrdersClassRepresentatives(k2);
```

[ [ 1, 1 ], [ 10, 1 ], [ 32, 1 ], [ 44, 1 ], [ 100, 1 ], [ 164, 1 ], [ 288, 1 ], [ 320, 1 ], [ 416, 1 ], [ 570, 1 ], [ 1046, 1 ], [ 1408, 1 ], [ 1792, 1 ], [ 2368, 1 ], [ 5632, 1 ] ]

Consequently $S_{12} < \Omega(10, 2)$ but $S_{12}$ is not simple group;

17. $A_n < \Omega^2(6, 4) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

18. $SL(2, 5) < SL(2, 9) < \Omega^2(4, 9) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

19. $GU(6, 2) < \Omega^2(12, 4) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

20. $SU(4, 2) < GL(5, 4) < \Omega^2(10, 4) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

21. $A\times S_n < SL(n, 2^2) < \Omega^2(2n, 2^2)$, A the direct product of cyclic group of order $a|q \pm 1, i > 1$, it is not our case, since our case $i = 1$;

22. If $\Omega(2n-1, q) < \Omega^2(2n, q) = G, q$ even, $n\geq 3$, then $\Omega(9, 2) < \Omega^2(10, 2)$, but $P\Omega(9, 2) \not< G$ since the irreducible 2-modular characters for $P\Omega(9, 2)$ by GAP:

```gap
gap> g:=GO(9,2);
```

$GO(0,9,2) 

```gap
gap> c:=CharacterTable("g");
```

Consequently if $PSL(9, 2) < P\Omega^2(10, 2)$ then it will be reducible;

23. $O^+(4, q) < \Omega^2(5, q) < \Omega^2(6, q) = G, q$ even, it is not our case, since our case $G = \Omega^2(10, 2)$;

24. HJ “Hall Janko group” $< \Omega^2(8, 4) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

25. $SU(4, 2) < \Omega^2(10, 2) = G$, but $PSU(4, 2) \not< G$ since the irreducible 2-modular characters for $PSU(4, 2)$ by GAP:

```gap
CharacterDegrees(CharacterTable("U4(2)")mod 2);
```

[ [ 1, 1 ], [ 4, 2 ], [ 6, 1 ], [ 14, 1 ], [ 20, 2 ], [ 64, 1 ] ]

Consequently if $PSU(4, 2) < P\Omega^2(10, 2)$ then it will be reducible;

26. $SU(4, 2) < \Omega^2(10, 4) = G$, it is not our case, since our case $G = \Omega^2(10, 2)$;

4.2 Primitive subgroups H of G which have the property that a minimal normal subgroup M of H which is not abelian is a finite primitive subgroup of rank three

A group $G$ has rank 3 in its permutation representation on the cosets of a subgroup $K$ if there are exactly 3 ($K, K'$)-double cosets and the rank of a transitive permutation group is the number of orbits of the stabilizer of a point, thus we may consider $P\Omega^2(10, 2)$ as group of permutations of the absolute points of the corresponding projective space, then $P\Omega^2(10, 2)$ is a transitive group of rank 3. In this section, we will consider the minimal normal subgroup $M$ of $H$ is not abelian and a finite primitive subgroup of rank three, so will use the classification of Kantor and Liebler {Result 4.2.1} for the primitive groups of rank three {see [3]}:
Result 4.2.1: If \( Y \) acts as a primitive rank 3 permutation group on the set \( X \) of cosets of a subgroup \( K \) of \( \text{Sp}(2n-2, q) \), \( \Omega^*(2n, q) \), \( \Omega(2n-1, q) \) or \( \text{SU}(n, q) \). Then for \( n \geq 3 \), \( Y \) has a simple normal subgroup \( M^* \), and \( M^* \subseteq Y \subseteq \text{Aut}(M^*) \), where \( M^* \) as follows:

\[
\begin{align*}
(\text{i}) & \quad M = \text{Sp}(4, q), \text{SU}(4, q), \text{SU}(5, q), \Omega^*(6, q), \Omega^*(8, q) \\
(\text{ii}) & \quad M = \text{SU}(n, 2), \Omega^*(2n, 2), \Omega^*(2n, 3) \quad \text{or} \quad \Omega(2n-1, 3), \\
(\text{iii}) & \quad M = \Omega(2n-1, 4) \quad \text{or} \quad \Omega(2n-1, 8), \\
(\text{iv}) & \quad M = \text{SU}(3, 3), \\
(\text{v}) & \quad \text{SU}(3, 5), \\
(\text{vi}) & \quad \text{SU}(4, 3), \\
(\text{vii}) & \quad \text{Sp}(6, 2), \\
(\text{viii}) & \quad \Omega(7, 3), \\
(\text{ix}) & \quad \text{SU}(6, 2).
\end{align*}
\]

The following Corollary is the main result of this section:

Corollary 4.2.2: If \( M \) is a non abelian simple group which is a finite primitive subgroup of rank three group of \( H \), then \( M \) is isomorphic to one of the following groups:

1. \( \text{PO}(6, 2) \);
2. \( \text{PO}^*(8, 2) \);

Proof: Let \( M \) is not an abelian finite primitive subgroup of rank three of \( H \), and \( M \) acts regularly on \( \Omega^*(2n, q) \) for \( \Omega^*(2n, q) \), \( \Omega(3, n) \) and \( \text{SU}(n, q) \). Consequently \( M \subseteq G \).

**CASE 1:** If \( M = \text{PSp}(4, q) \), but in our case \( q = 2 \), then \( M = \text{PSp}(4, 2) \) But \( \text{PSp}(4, 2) \) is not simple, Consequently \( M \not\subseteq G \).

**CASE 2:** If \( M = \text{PSU}(4, q) \), but in our case \( q = 2 \), then \( M = \text{PSU}(4, 2) \), but the irreducible 2-modular characters for \( \text{PSU}(4, 2) \) by GAP:

\[
gap> \text{gap}\text{-characters} \text{CharacterTable}("U4(2)"	ext{-mod} 2); \\
\quad [ [ 1, 1 ], [ 4, 2 ], [ 6, 1 ], [ 14, 1 ], [ 20, 2 ], [ 64, 1 ] ].
\]

and none of these of degree 10. Consequently \( M \not\subseteq G \).

**CASE 3:** If \( M = \text{PSU}(5, 2) \), then \( M \) is generated by transvections and from Corollary 4.1.3, we get \( M \not\subseteq G \).

**CASE 4:** If \( M = \text{PO}(6, 2) \), then the irreducible characters for \( \text{PO}(6, 2) \) by GAP are:

\[
gap> \text{gap}\text{-characters} \text{OrdersClassRepresentatives}(\text{c}); \\
\quad [ 1, 2, 4, 2, 4, 2, 8, 4, 4, 4, 3, 6, 12, 5, 10, 20, 2, 2, 4, 4, 8, 8, 8, 6, 6 ]
\]

Consequently \( M \subseteq G \).

**CASE 5:** If \( M = \text{PO}^*(8, 2) \), then the irreducible characters for \( \text{PO}^*(8, 2) \) by GAP are:

\[
gap> \text{gap}\text{-characters} \text{OrdersClassRepresentatives}(\text{c}); \\
\quad [ 1, 2, 4, 4, 2, 4, 2, 8, 4, 4, 4, 3, 6, 12, 5, 10, 20, 2, 2, 4, 4, 8, 8, 8, 6, 6 ]
\]

Consequently \( M \subseteq G \).

**CASE 6:** if \( M = \text{PSU}(n, 2) \), and in our case \( n = 5 \), thus \( M = \text{PSU}(5, 2) \), then by case 3 we get \( M \not\subseteq G \).

**CASE 7:** If \( M = \text{PO}^*(2n, 2) \), then in our case \( n = 5 \), thus we need to consider \( \text{PO}^*(10, 2) \):

- If \( M = \text{PO}^*(10, 2) \), then \( M \) is an improper subgroup of \( G \), but we search for a proper subgroup of \( G \).
- If \( M = \text{PO}^*(10, 2) \), then \( \text{gap}\text{-characters} \text{OrdersClassRepresentatives}(\text{c}); \\
\quad [ 1, 2, 4, 4, 2, 4, 2, 8, 4, 4, 4, 3, 6, 12, 5, 10, 20, 2, 2, 4, 4, 8, 8, 8, 6, 6 ]
\]

Consequently \( M \subseteq G \).

**CASE 8:** If \( M = \text{PO}(2n, 3) \), then in our case \( n = 5 \), thus we need to consider \( \text{PO}^*(10, 3) \):

- \( \text{PO}(2n, q) \), \( n \geq 4 \), has no projective representation in \( G \) of degree less than \( q^{n-1}-1 \). \{see [12] and [13]\}, when \( q = 3, n \geq 4 \), this bound is greater than 10, thus \( \text{PO}^*(10, 3) \not\subseteq G \).
- \( \text{PO}(2n, q) \), \( n \geq 4 \), has no projective representation in \( G \) of degree less than \( q^{n-1}+1 \). \{see [12] and [13]\}, when \( q = 3, n \geq 4 \), this bound is greater than 10, thus \( \text{PO}^*(10, 3) \not\subseteq G \).

**CASE 9** If \( M = \text{PO}(2n-1, 3) \), then in our case \( n = 5 \), thus, we need to consider \( M = \text{PO}(9, 3) \): But \( \text{PO}(2n+1, q) \), \( n \geq 3 \),
has no projective representation in G of degree less than \(q^{n-1} - 1\), \{see [12] and [13]\}, when \(q = 3, n \geq 3\), this bound is greater than 10, thus \(PΩ(9, 3) \not\subset G\).

**CASE 10:** If \(M = PΩ(2n-1, 4)\), then in our case \(n = 5\), thus, we need to consider \(M = PΩ(9, 4)\), but \(PΩ(2n+1, q)\), \(n \geq 3\), has no projective representation in G of degree less than \(q^{n-1} - 1\), \{see [12] and [13]\}, when \(q = 4, n \geq 3\), this bound is greater than 10, thus \(PΩ(9, 4) \not\subset G\).

**CASE 11:** If \(M = PΩ(2n-1, 8)\), then in our case \(n = 5\), thus, we need to consider \(M = PΩ(9, 8)\), but \(PΩ(2n+1, q)\), \(n \geq 3\), has no projective representation in G of degree less than \(q^{n-1} - 1\), \{see [12] and [13]\}, when \(q = 4, n \geq 3\), this bound is greater than 10, thus \(PΩ(9, 8) \not\subset G\).

**CASE 12:** If \(M = PSU(3, 3)\), then the irreducible 2-modular characters for \(PSU(3, 3)\) by GAP are:

```gap
gap> CharacterDegrees(CharacterTable("U3(3)")mod 2);
[ [ 1, 1 ], [ 6, 1 ], [ 14, 1 ], [ 32, 2 ] ],
```

and none of these of degree 10. Thus \(M \not\subset G\).

**CASE 13:** If \(M = PSU(3, 5)\), then the irreducible 2-modular characters for \(PSU(3, 5)\) by GAP are:

```gap
gap> CharacterDegrees(CharacterTable("U3(5)")mod 2);
[ [ 1, 1 ], [ 20, 1 ], [ 34, 1 ], [ 70, 1 ], [ 120, 1 ], [ 112, 2 ] ]
```

And none of these of degree 10. Thus \(M \not\subset G\).

**CASE 14:** If \(M = PSU(4, 3)\), then the irreducible 2-modular characters for \(PSU(4, 3)\) by GAP are:

```gap
gap> CharacterDegrees(CharacterTable("U4(3)")mod 2);
[ [ 1, 1 ], [ 20, 1 ], [ 34, 2 ], [ 70, 4 ], [ 120, 1 ], [ 640, 2 ], [ 896, 1 ] ]
```

And none of these of degree 10. Thus \(M \not\subset G\).

**CASE 15:** If \(M = PSp(6, 2)\), then the irreducible 2-modular characters for \(PSp(6, 2)\) by GAP are:

```gap
gap> CharacterDegrees(CharacterTable("S6(2)")mod 2);
[ [ 1, 1 ], [ 6, 1 ], [ 8, 1 ], [ 14, 1 ], [ 48, 1 ], [ 64, 1 ], [ 112, 1 ], [ 512, 1 ] ]
```

And none of these of degree 10. Thus \(M \not\subset G\).

**CASE 16:** If \(M = PΩ(7, 3)\), then \(PΩ(2n+1, q)\), \(n \geq 3\), has no projective representation in G of degree less than \(q^{n-1} - 1\), \{see [12] and [13]\}, when \(q = 3, n \geq 3\), this bound is greater than 10, thus \(M \not\subset G\).

**CASE 17:** If \(M = PSU(6, 2)\), then the irreducible 2-modular characters for \(PSU(6, 2)\) by GAP are:

```gap
gap> CharacterDegrees(CharacterTable("U6(2)")mod 2);
[ [ 1, 1 ], [ 20, 1 ], [ 34, 1 ], [ 70, 2 ], [ 154, 1 ], [ 400, 1 ], [ 896, 2 ], [ 1960, 1 ], [ 3114, 1 ], [ 32768, 1 ] ]
```

And none of these of degree 10. Thus \(M \not\subset G\).

Now, we will determine the maximal primitive group of \(C_5\):

**Theorem 4.2:** There is no maximal primitive subgroup of \(G\) which has the property that a minimal normal subgroup \(M\) of \(H\) is not abelian group.

**Proof:**

We will prove this theorem by finding the normalizers \(N\) of the groups of Corollary 4.1 and determine which of them are maximal:

From [14], the normalizer of \(Sp(2n, k)\) in \(SL(2n, k)\) is \(SGSp(2n, k) = GSp(2n, k) \cap SL(2n, k)\).

From [15], the normalizer of \(SU(n, k)\) in \(SL(n, k)\) is \(SGU(n, k) = GU(n, k) \cap SL(n, k)\). From [16], the normalizer of \(SO(n, k)\) in \(SL(n, k)\) is \(SGO(n, k) = GO(n, k) \cap SL(n, k)\). Thus,

- If \(Y = PSL(5, 2)\), then \(N = PGL(5, 2)\) \{see [5]\} but \(PGL(5, 2)\) is a subgroup of \(H_5\), thus \(Y\) is not a maximal subgroup of \(G\).
- If \(Y = PSp(5, 2)\), then \(N = PSGSp(5, 2)\) but in \(G\), \(PSGSp(5, 2) = PSp(5, 2)\), in this case \(Y\) is a subgroup of \(H_5\), thus \(Y\) is not a maximal subgroup of \(G\).
- If \(Y = PO^+ (5, 2)\), then \(N = PSGO^+(5, 2)\), which is not a maximal in \(G\) since \(PSGO^+(5, 2)\) is a subgroup of \(H_5\).
- If \(Y = PO^-(5, 2)\), then \(N = PSGO^-(5, 2)\), which is not a maximal in \(G\) since \(PSGO^-(5, 2)\) is a subgroup of \(H_5\).
• If $Y = P\Omega(6, 2)$, then $N = PSGO(6, 2)$, which is not a maximal in $G$ since $PSGO(6, 2)$ is a subgroup of $H_6$.

• If $Y = P\Omega(8, 2)$, then $N = PSGO(8, 2)$, which is not a maximal in $G$ since $PSGO(8, 2)$ is a subgroup of $H_{10}$.

This completes the proof of theorem 1.1.

REFERENCES


