Lexicographic Non-cooperative Game’s Mixed Extension with Criteria

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ABSTRACT

In this paper the new concept of mixed extensional finite lexicographic no cooperative game \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) of finite dimension is discussed. In this extension the players choose the components \( \Gamma^1, \ldots, \Gamma^r \) of the game \( \Gamma \) with given probabilistic distributions on the set of such scalar game-criteria. The concept of situation, a situation in the coordinated strategies, an equilibrium situation with positive weights of preferences are given in a lexicographic game. And conditions of the existence of such equilibrium situation are proved in a lexicographic no cooperative game.

Keywords: lexicographic game, non cooperative game, strategy, equilibrium situation.

1. INTRODUCTION

Consider a finite lexicographic non cooperative \( n \) person’s game \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) of \( r \)-dimensional [1]:

\[
\Gamma = \langle I, \{\mathcal{X}_i\}_{i \in I}, \{H_i\}_{i \in I} \rangle = (\Gamma^1, \ldots, \Gamma^r),
\]

where \( I = \{1, 2, \ldots, n\} \) - is set of players,
\( \mathcal{X}_i = \{1, \ldots, m_i\} \) - \( i \in I \) player’s set of strategies,
\( H_i = (H^1_i, \ldots, H^r_i) \) - is a payoff function of \( i \in I \) player on \( \mathcal{X} = \prod_{i \in I} \mathcal{X}_i \) the set of situation, that has the meanings in \( \mathbb{R}^r \) space and these vectorial meanings in the situation are ranked strongly according to their importance with necessity of \( r \) criteria, i.e. they are lexicographically ranked with \( \geq^L \) or \( >^L \) relation.Lexicographic strong preference \( a >^L b \) for \( a = (a_1, \ldots, a_r) \) and \( b = (b_1, \ldots, b_r) \) vectors means that if fulfills one of the following \( r \) conditions: 1. \( a_1 > b_1 \), 2. \( a_1 = b_1, a_2 > b_2, \ldots, r : a_1 = b_1, \ldots, a_{r-1} = b_{r-1}, a_r > b_r \) and \( a >^L b \), if \( a >^L b \) or \( a = b \). This definition means, that scalar criteria \( H^1, \ldots, H^r \) are related to criteria \( \Gamma^1, \ldots, \Gamma^r \) of \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) game, at this time \( H^1, \ldots, H^r \) are strong ranking with decrease preference. Game (1) is also called a non cooperative multicriteria game strictly ordered with necessity of criteria. It is known that not in every lexicographic non cooperative game \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) exists equilibrium situation both in pure and mixed strategies [1].

The finite lexicographic antagonistic game is defined by P.C.Fishburn [2] and V.V.Podinovski [3]. They have proved that in such kind of games Nash’s equilibrium may exist neither in the pure nor in the mixed strategies. Conditions of the existence of Nash’s equilibrium in \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) lexicographic non cooperative game is studied by G.N.Beltadze [4,5,6]. The full analysis of one class of dyadic lexicographic games is studied by M. Salukvadze, G. Beltadze and F. Criado [7].

In this paper we introduce a new concept of mixed extension when players select at random \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) game’s \( \Gamma^1, \ldots, \Gamma^r \) game-criteria. Players can negotiate in advance about the probability distribution of \( \Gamma^1, \ldots, \Gamma^r \) game’s set of criteria or define them with these criteria’s own preference.

2. THE MAIN PART

In \( \Gamma = (\Gamma^1, \ldots, \Gamma^r) \) game players’ mixed strategies is the probability distribution on their set of pure strategies. Namely, suppose \( X_i \) is a any kind of mixed strategy of \( i \in I \) player and a probability, that \( X_i \) distribution assigns to a concrete pure \( l_i (l_i = 1, \ldots, m_i) \)
strategy, note by \(X_i(l_i)\). \(i \in I\) Player’s every mixed strategies note by \(\Sigma_i\).

Suppose, each \(i \in I\) player use it’s mixed \(X_i\) strategy. Every \(1, \ldots, n\) player’s mixed strategies, as a probability distributions are independent, i.e. probability appearance for every \(I = (l_1, \ldots, l_n) \in \chi = \prod_{i=1}^n \chi_i\) situation is the probabilistic product \(X_1(l_1), \ldots, X_n(l_n)\) of choosing its component strategies. Thus, in \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) game probabilistic distribution \(X\) on every sets of situation \(\chi = \prod_{i=1}^n \chi_i\) is given with relation \(X(I) = X(l_1, \ldots, l_n) = X_1(l_1) \times \ldots \times X_n(l_n)\) for every \(I = (l_1, \ldots, l_n) \in \chi\) situation.

Player’s payoff functions in a \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) game’s we write as following

\[
H_i(X) = (H_1^i(X), \ldots, H_n^i(X)) = \left( \sum_{l_i \in \chi_i} H_i^1(l_1, \ldots, l_n)X_1(l_1) \times \ldots \times X_n(l_n) \right)
\]

\[
\sum_{l_i \in \chi_i} H_i^2(l_1, \ldots, l_n)X_2(l_2) \times \ldots \times X_n(l_n) \right), \quad \sum_{l_i \in \chi_i} H_i^3(l_1, \ldots, l_n)X_3(l_3) \times \ldots \times X_n(l_n) \right),
\]

\[
\sum_{l_i \in \chi_i} H_i^{n}(l_1, \ldots, l_n)X_n(l_n) \right),
\]

\[
\forall l_i \in \chi_i, \quad (H_1^i(X), \ldots, H_n^i(X)) \geq (H_1^i(X), \ldots, H_n^i(X))
\]

\[
\sum_{l_i \in \chi_i} H_i^{n}(l_1, \ldots, l_n)X_n(l_n) \right), \quad \sum_{l_i \in \chi_i} H_i^{n}(l_1, \ldots, l_n)X_n(l_n) \right),
\]

\[
\forall l_i \in \chi_i
\]

As we have noted above, the set \(\sigma(\Gamma)\) can be empty - \(\sigma(\Gamma) = \emptyset\).

We introduce another version of the mixed extension of lexicographic non cooperative \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) game. In the given game of \(\Gamma^k (k = 1, \ldots, r)\) criterion the selection of probability for the \(i \in I\) player note as \(p_i^k (k = 1, \ldots, r)\). Thus

\[
p_i^k \geq 0 \geq (k = 1, \ldots, r; i = 1, \ldots, n) \quad \text{and} \quad \sum_{k=1}^r p_i^k = 1.
\]

Note that \(p_i = (p_i^1, \ldots, p_i^r) \in \prod_{i}, \quad i \in I\). Also, with the importance of \(\Gamma^k (k = 1, \ldots, r)\) game we mean that \(p_i^1 > p_i^2 > \ldots > p_i^r\).

We suppose that the selection of \(\Gamma^k (k = 1, \ldots, r)\) criteria in \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) game places take independently from the selection of pure and mixed strategies.

In conditions of such \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) lexicographic game the \(i \in I\) player’s strategy is pair \(S_j = (X_i, P_j)\). Such kind of lexicographic non cooperative \(\Gamma = (\Gamma^1, \ldots, \Gamma^n)\) game, where the situation \(S = (S_1, \ldots, S_n)\), note by \(\Gamma(S)\). In game \(\Gamma(S)\) players’ payoffs in the situation \(S = (S_1, \ldots, S_n)\) are defined as the following way:

\[
H_i(S) = H_i((X_1, P_1), \ldots, (X_n, P_n)) = (p_1^1 \ldots p_n^1 \sum_{l_i \in \chi_i} H_i^1(I)X(I), \ldots, p_1^r \ldots p_n^r \sum_{l_i \in \chi_i} H_i^r(I)X(I)),
\]

\(i\in I\).

Definition 1. The situation \(S^* = (S_1^*, \ldots, S_n^*) = ((X_1^*, P_1^*), \ldots, (X_n^*, P_n^*)) \in \sigma(\Gamma(S^*))\), if any \(I_i \in \chi_i, i\in I\) takes place the following inequality:

\[
(H_1^i(X_1^*, P_1^*), \ldots, H_n^i(X_n^*, P_n^*)) \geq (p_1^i \ldots p_n^i)
\]

\[
\sum_{l_i \in \chi_i} H_i^1(l_1, \ldots, l_n)X_1(l_1) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^2(l_1, \ldots, l_n)X_2(l_2) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^3(l_1, \ldots, l_n)X_3(l_3) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^n(l_1, \ldots, l_n)X_n(l_n),
\]

\(\forall l_i \in \chi_i\),

\[
(3)
\]

\[
(H_1^i(X_1^*, P_1^*), \ldots, H_n^i(X_n^*, P_n^*)) \geq (p_1^i \ldots p_n^i)
\]

\[
\sum_{l_i \in \chi_i} H_i^1(l_1, \ldots, l_n)X_1(l_1) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^2(l_1, \ldots, l_n)X_2(l_2) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^3(l_1, \ldots, l_n)X_3(l_3) \times \ldots \times X_n(l_n),
\]

\[
\sum_{l_i \in \chi_i} H_i^n(l_1, \ldots, l_n)X_n(l_n),
\]

\(\forall l_i \in \chi_i\).
Definition 2. 2.1. The quantity \( p_1^k p_2^k ... p_n^k \) (\( k = 1,...,r \)) is called the weights of preferences;

2.2. We say that the strategies \( S_1,...,S_n \) are coordinated, if any \( k = 1,...,r \) or \( p_1^k p_2^k ... p_n^k > 0 \), or \( p_1^k = ... = p_n^k = 0 \);

2.3. We call \( P_i = (p_i^1,...,p_i^r), i \in I \) - the situation in coordinative strategies, and \( (P_1,...,P_n) \) - the situation in coordinative strategies;

2.4. If \( (S_1^*,...,S_n^*) \in \sigma(\Gamma(S^*)) \) and every \( p_1^k p_2^k ... p_n^k > 0 \) (\( k = 1,...,r \)), then we say that \( (S_1^*,...,S_n^*) \) is the equilibrium situation with positive weights of preferences, and if some \( p_1^k = ... = p_n^k = 0 \), then we say that \( (S_1^*,...,S_n^*) \) is the equilibrium situation with nonnegative weights of preferences.

Lemma. For all positive numbers \( \alpha_i > 0, \alpha_r > 0 \) the following inequalities are equivalent

\[
(a) \quad (a_1,...,a_r) \leq (b_1,...,b_r),
\]

\[
(b) \quad (\alpha_1 a_1,...,\alpha_r a_r) \leq (\alpha_1 b_1,...,\alpha_r b_r).
\]

Proof. The proof follows from the definition of \( L \). Suppose we have \( (a) \), i.e. one of the following conditions takes place:

1. \( a_1 < b_1 \);
2. \( a_i = b_i, \quad a_2 \leq b_2; \quad \ldots ; \quad r, \quad a_k = b_k \) (\( k = 1,...,r-1 \)).

For positive \( \alpha_1,...,\alpha_r \) follows that

\[
(a') \quad (a_1,\ldots,a_r) \leq (b_1,\ldots,b_r),
\]

\[
(b') \quad (\alpha_1 a_1,\ldots,\alpha_r a_r) \leq (\alpha_1 b_1,\ldots,\alpha_r b_r).
\]

From these conditions follows \( (b') \). Conversely, from \( (b') \) follows \( (a') \) and therefore fulfills \( (a) \).

Theorem. If \( (S_1^*,...,S_n^*) \in \sigma(\Gamma(S^*)) \) with the positive weights of preferences, then

\[
X^* = (X_1^*,...,X_n^*) \in \sigma(\Gamma) \quad \text{and vice versa, if} \quad X^* = (X_1^*,...,X_n^*) \in \sigma(\Gamma), \quad \text{then there is a situation in coordinated strategies} \quad (P_1^*,...,P_n^*), \quad \text{that}
\]

\[
((X_1^*,P_1^*),...,X_n^*,P_n^*) \in \sigma(\Gamma(S^*)�).
\]

Proof. According to the condition of the theorem we have \( (3) \), i.e. we have the inequalities

\[
(p_1^* ... p_n^* \sum_{l^i \in X} H_1^i(l)x^*(l),... , \sum_{l^i \in X} H_n^i(l)x^*(l) \geq \lbrack \sum_{l^i \in X} H_1^i(l)x^*(l) \times \ldots \times \sum_{l^i \in X} H_n^i(l)x^*(l) \rbrack \forall l^i \in X^*. \]

Let set such positive numbers \( p_1^k > p_2^k > ... > p_n^k > 0 \) (\( k = 1,...,r \)), that

\[
\sum p_i^k = 1. \quad \text{Note} \quad P_i^* = (p_i^1,...,p_i^r), \quad i \in I. \quad \text{Then on the strength of lemma from (4) we have (3). This means,}
\]

\[
((X_1^*,P_1^*),...,X_n^*,P_n^*) \in \sigma(\Gamma(S^*)�).
\]

The theorem is proved.
As we have noted above, vectors $P_1^*, ..., P_n^*$ can be defined by coordination or with their own preferences related to $\Gamma^1, ..., \Gamma^r$. These vectors must express the weights of given criteria in a certain way. In order to find such weights we can use the method of T. Saaty about hierarchical analysis necessities of mutually comparisons of $\Gamma^1, ..., \Gamma^r$ games.

Some variants are possible. In the case of players’ objective behavior, any game $\Gamma^k (k = 1, ..., r - 1)$ must be absolutely preferable to $\Gamma^{k+1}$, that in the quantitative estimates of T. Saaty expressed through $9$.

Let find the weight vector of the criteria $\Gamma^1, ..., \Gamma^r$ for the first player [8]:

$$S_1 = \begin{bmatrix}
\Gamma^1 \\
\Gamma^2 \\
\Gamma^3 \\
\vdots \\
\Gamma^r \\
\end{bmatrix} = \begin{bmatrix}
1 & 9 & 9^2 & \ldots & 9^{r-2} & 9^{r-1} \\
1 & 1 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \ldots & \ldots & \ldots & \ldots & 1
\end{bmatrix}
$$

Here $\alpha^0 = (9^{r-1}, 9^{r-2}, ..., 9, 1)$, by normalizing we have $P_1^*$ vector.

For example, if $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$, then

$$S_1 = \begin{bmatrix}
\Gamma^1 \\
\Gamma^2 \\
\Gamma^3 \\
\end{bmatrix} = \begin{bmatrix}
1 & 9 & 81 \\
1 & 9 \\
1
\end{bmatrix}
$$

And $\alpha^0 = (81, 9, 1)$. Since $81 + 9 + 1 = 91$, therefore $P_1^* = (81/91, 9/91, 1/91)$, that is the same as $P_2^*$ and $P_3^*$.

Apart from this case, players can discuss the variants of analyses of preference on the set $\Gamma^1, ..., \Gamma^r$. For example, one player can suppose that $\Gamma^1$ is preferable to $\Gamma^3$ very strongly (by estimating $7$) or discuss other subjective demonstrative estimations.

Note. With the help of playing mechanism of $(p_1^*, ..., p_n^*)$ vector will be chosen scalar non cooperative game from $\Gamma^1, ..., \Gamma^r$, where always exists the equilibrium situation.

3. CONCLUSION

For discussing any $\Gamma = (\Gamma^1, ..., \Gamma^r)$ lexicographic game and for the existence of the optimal strategies for the players in this game we reasonably consider that from scalar components $\Gamma^1, ..., \Gamma^r$ is the only optimal choice and finding the equilibrium situation in it. Therefore a new mixed distribution of a lexicographic non cooperative $\Gamma = (\Gamma^1, ..., \Gamma^r)$ game means having a player’s probability distributions respectively $P_1^*, ..., P_n^*$ on the set of $\Gamma^1, ..., \Gamma^r$-game criteria. With the help $P_1^*, ..., P_n^*$ the corresponding weight $p_1^* \ldots p_n^*$, $p_1^* \ldots p_n^*$ coefficients of $\Gamma^1, ..., \Gamma^r$ games are defined. $P_1^*, ..., P_n^*$ vectors can be defined by means of players’ coordination or with $\Gamma^1, ..., \Gamma^r$ games’ own preferences. These vectors express the given criteria’s weights. In order to find them we use T. Saaty’s method of hierarchical analyses.’

REFERENCES


